

# GEOMETRIC PROPERTIES OF log-POLYHARMONIC MAPPINGS

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**ABSTRACT.** In this paper, a class of log-polyharmonic mappings  $\mathcal{L}_p\mathcal{H}$  together with its subclass  $\mathcal{L}_p\mathcal{H}(G)$  in the unit disk  $\mathbb{D} = \{z : |z| < 1\}$  is introduced, and several geometrical properties such as the starlikeness, convexity and univalence are investigated. In particular, we consider the Goodman-Saff conjecture and prove that the conjecture is true in  $\mathcal{L}_p\mathcal{H}(G)$ .

## 1. INTRODUCTION

A complex-valued mapping  $f$  defined in a domain  $D \subset \mathbb{C}$  is called *harmonic* if  $\Delta f = 0$ , where  $\Delta$  represents the complex Laplacian operator

$$\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

It is known that any harmonic mapping  $f$  in a simply connected domain  $D$  can be written in the form  $f = g + \bar{h}$ , where  $g$  and  $h$  are analytic (cf. [16]). A complex-valued mapping  $F$  in a domain  $D$  is *biharmonic* if the Laplacian of  $F$  is harmonic, that is,  $F$  satisfies the biharmonic equation  $\Delta(\Delta F) = 0$ . It can be shown that in a simply connected domain  $D$ , every biharmonic mapping has the form

$$F(z) = G_1(z) + |z|^2 G_2(z),$$

where both  $G_1$  and  $G_2$  are harmonic in  $D$ .

More generally, a  $2p$  ( $p \geq 1$ ) times continuously differentiable complex-valued mapping  $F$  in a domain  $D \subset \mathbb{C}$  is *polyharmonic* if  $F$  satisfies the polyharmonic equation  $\Delta^p F = \Delta(\Delta^{p-1} F) = 0$ . In a simply connected domain  $D$ , a mapping  $F$  is polyharmonic if and only if  $F$  has the following representation:

$$(1.1) \quad F(z) = \sum_{k=1}^p |z|^{2(k-1)} G_k(z),$$

where each  $G_k$  is harmonic, i.e.,  $\Delta G_k(z) = 0$  for each  $k \in \{1, \dots, p\}$  (cf. [7, 8]).

Obviously, when  $p = 1$  (resp. 2),  $F$  is harmonic (resp. biharmonic). The biharmonic equation arises when modeling and solving problems in a number of areas of science and engineering. Most importantly, it is encountered in plane problems of elasticity and is used to describe creeping flows of viscous incompressible fluids. For instance, the determination of stress states around cavities in the stressed elastic body, regardless of cavity shapes, in its analytical approach has to be based on

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selection of a stress function that will satisfy the biharmonic equation (for details, see [18, 19, 21]). However, the investigation of biharmonic mappings in the context of geometric function theory is a recent one (cf. [2, 4, 5, 13]). The reader is referred to [7, 8, 10, 11, 12] for further discussions on polyharmonic mappings and [9, 14, 16] as well as the references therein for the properties of harmonic mappings.

A mapping  $G$  is said to be *log-harmonic* in a domain  $D$  if there is an analytic function  $a$  such that  $G$  is a solution of the nonlinear elliptic partial differential equation

$$\frac{\overline{G_z}}{G} = a \frac{G_z}{G}.$$

It has been shown that if  $G$  is a nonvanishing log-harmonic mapping, then  $G$  can be expressed as  $G = f\overline{h}$ , where both  $f$  and  $h$  are analytic. A complex-valued mapping  $F$  in a domain  $D$  is *log-biharmonic* if  $\log F$  is biharmonic, that is, the Laplacian of  $\log F$  is harmonic. Further, we say that  $F$  is *log-polyharmonic* if  $\log F$  is polyharmonic. It can be easily shown that every log-polyharmonic mapping  $F$  in a simply connected domain  $D$  has the form

$$(1.2) \quad F(z) = \prod_{k=1}^p G_k(z)^{|z|^{2(k-1)}},$$

where all  $G_k$  are nonvanishing log-harmonic mappings in  $D$  for  $k \in \{1, \dots, p\}$ . When  $p = 1$  (resp.  $p = 2$ ), log-polyharmonic  $F$  is log-harmonic (resp. log-biharmonic) (cf. [22, 23]).

Some physical problems are modeled by log-biharmonic mappings, particularly, those arising in fluid flow theory and elasticity. The log-biharmonic mappings are closely associated with the biharmonic mappings, which appear in Stokes flow problems etc. Recently, the properties of log-harmonic and log-biharmonic mappings have been investigated by a number of authors (cf. [1, 3, 6, 20, 24]).

We say that a univalent polyharmonic mapping  $F$  defined in  $\mathbb{D}$ , with  $F(0) = 0$ , is *starlike* if the curve  $F(re^{it})$  is starlike with respect to the origin for each  $0 < r < 1$ . In other words,  $F$  is starlike if

$$\frac{\partial}{\partial t}(\arg F(re^{it})) = \operatorname{Re} \frac{zF_z(z) - \overline{z}F_{\overline{z}}(z)}{F(z)} > 0$$

for  $z \in \mathbb{D} \setminus \{0\}$ .

A univalent polyharmonic mapping  $F$  defined in  $\mathbb{D}$ , with  $\frac{\partial}{\partial t}F(re^{it}) \neq 0$  whenever  $0 < r < 1$ , is said to be *convex* if the curve  $F(re^{it})$  is convex for each  $0 < r < 1$ . In other words,  $F$  is convex if

$$\frac{\partial}{\partial t} \left( \arg \frac{\partial}{\partial t} F(re^{it}) \right) > 0$$

for  $z \in \mathbb{D} \setminus \{0\}$ .

Throughout this paper, we shall discuss log-polyharmonic mappings defined in  $\mathbb{D}$ . We use  $J_F$  to denote the Jacobian of  $F$ , that is,

$$J_F = |F_z|^2 - |F_{\overline{z}}|^2.$$

It is known that  $F$  is sense-preserving and locally univalent if  $J_F > 0$ . For convenience, we introduce the following notations:

$$\mathcal{L}_p\mathcal{H} = \{F : F \text{ has the expression (1.2) in } \mathbb{D}\}$$

and

$$\mathcal{L}_p\mathcal{H}(G) = \left\{ F : F(z) = f(z)h(\bar{z}) \prod_{k=1}^p G(z)^{\lambda_k |z|^{2(k-1)}} \right\},$$

where  $G$  is a non-vanishing log-harmonic mapping in  $\mathbb{D}$ ,  $f$  and  $h$  are non-vanishing analytic functions in  $\mathbb{D}$ , and all  $\lambda_k$  are complex constants.

Obviously, we have the following.

**Proposition 1.** *Each element in  $\mathcal{L}_p\mathcal{H}$  is log-polyharmonic, and  $\mathcal{L}_p\mathcal{H}(G) \subset \mathcal{L}_p\mathcal{H}$ .*

The main aim of this paper is two-fold. First, we discuss several geometrical properties such as the starlikeness, convexity and univalence for elements in  $\mathcal{L}_p\mathcal{H}(G)$ . Our results consist of four theorems which will be stated and proved in Section 2. Second, we consider the Goodman-Saff conjecture (cf. [25]). Our result, Theorem 5 in Section 3, shows that the answer to this conjecture is positive in  $\mathcal{L}_p\mathcal{H}(G)$ . This result will be demonstrated in Section 3.

## 2. PROPERTIES OF LOG-POLYHARMONIC MAPPINGS

First, we define a linear operator  $\mathcal{L}$  by

$$\mathcal{L} = z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}}.$$

The definition leads to the following two properties:

- $\mathcal{L}[\alpha f + \beta g] = \alpha \mathcal{L}[f] + \beta \mathcal{L}[g]$ ,
- $\mathcal{L}[fg] = \mathcal{L}[f]g + f\mathcal{L}[g]$ ,

where  $f, g$  are  $C^1$  mappings and  $\alpha, \beta$  are complex constants.

**Theorem 1.** *Suppose  $F \in \mathcal{L}_p\mathcal{H}$  with  $F(z) = \prod_{k=1}^p G_k(z)^{|z|^{2(k-1)}}$ . Then*

- (1)  $\mathcal{L}[\log F(z)] = \sum_{k=1}^p |z|^{2(k-1)} \mathcal{L}[\log G_k(z)]$ ; and
- (2)  $\mathcal{L}^n[\log F(z)] = \sum_{k=1}^p |z|^{2(k-1)} \mathcal{L}^n[\log G_k(z)]$ ,

where  $n \geq 2$  is an integer.

*Proof.* A simple calculation yields

$$\log F(z) = \sum_{k=1}^p |z|^{2(k-1)} \log G_k(z),$$

and for all  $k \in \{1, \dots, p\}$ ,

$$\mathcal{L}[|z|^{2(k-1)}] = 0.$$

Using the product rule property and the linearity of the operator  $\mathcal{L}$ , we have

$$\mathcal{L}[\log F(z)] = \sum_{k=1}^p \mathcal{L}[|z|^{2(k-1)} \log G_k(z)] = \sum_{k=1}^p |z|^{2(k-1)} \mathcal{L}[\log G_k(z)],$$

which shows that the first statement in the theorem is true. The proof for the second statement easily follows from the first one and mathematical induction.  $\square$

By replacing each  $G_k$  with  $G^{\lambda_k}$  in Theorem 1, we have the following corollary.

**Corollary 1.** *Suppose  $F \in \mathcal{L}_p\mathcal{H}(G)$  with*

$$F(z) = \prod_{k=1}^p G(z)^{\lambda_k |z|^{2(k-1)}},$$

where  $\mathcal{L}[\log G(z)] \neq 0$  and  $\sum_{k=1}^p \lambda_k |z|^{2(k-1)} \neq 0$ . Then

$$\frac{\mathcal{L}^n[\log F(z)]}{\mathcal{L}[\log F(z)]} = \frac{\mathcal{L}^n[\log G(z)]}{\mathcal{L}[\log G(z)]}, \quad n \geq 2.$$

From the definition of starlikeness, we know that

$$F(z) = \prod_{k=1}^p G(z)^{\lambda_k |z|^{2(k-1)}} \in \mathcal{L}_p\mathcal{H}(G)$$

is starlike if and only if  $F(0) = 0$ ,  $F$  is univalent and  $\operatorname{Re} \frac{\mathcal{L}[F(z)]}{F(z)} > 0$  for  $z \in \mathbb{D} \setminus \{0\}$ . Since  $G$  is a non-vanishing log-harmonic mapping in  $\mathbb{D}$ , then it is impossible for  $F$  to reach this requirement. Instead, we consider the starlikeness of  $\log F$  and  $\log G$ .

**Theorem 2.** *Suppose that  $F \in \mathcal{L}_p\mathcal{H}(G)$  with*

$$F(z) = \prod_{k=1}^p G(z)^{\lambda_k |z|^{2(k-1)}},$$

*$\log F$  and  $\log G$  are both univalent, and  $\sum_{k=1}^p \lambda_k |z|^{2(k-1)} \neq 0$ . Then  $\log G$  is starlike if and only if  $\log F$  is starlike.*

*Proof.* Obviously, for  $z \in \mathbb{D} \setminus \{0\}$ ,

$$\begin{aligned} \operatorname{Re} \left( \frac{z(\log F(z))_z - \bar{z}(\log F(z))_{\bar{z}}}{\log F(z)} \right) &= \operatorname{Re} \frac{\sum_{k=1}^p \lambda_k |z|^{2(k-1)} (z(\log G(z))_z - \bar{z}(\log G(z))_{\bar{z}})}{\sum_{k=1}^p \lambda_k |z|^{2(k-1)} \log G(z)} \\ &= \operatorname{Re} \frac{z(\log G(z))_z - \bar{z}(\log G(z))_{\bar{z}}}{\log G(z)}. \end{aligned}$$

Then the assumptions imply that  $\log G$  is starlike if and only if  $\log F$  is starlike.  $\square$

To discuss the local univalence of  $\log F$ , where  $F \in \mathcal{L}_p\mathcal{H}(G)$ , the following lemma is useful.

**Lemma 1.** Suppose  $F \in \mathcal{L}_p \mathcal{H}(G)$  with  $F(z) = f(z)h(\bar{z}) \prod_{k=1}^p G(z)^{\lambda_k |z|^{2(k-1)}}$ . Then the Jacobian of  $\log F$ ,  $J_{\log F}$ , is given by

$$\begin{aligned} J_{\log F}(z) &= \left| \frac{f'(z)}{f(z)} \right|^2 - \left| \frac{h'(\bar{z})}{h(\bar{z})} \right|^2 + |B(z)|^2 J_{\log G}(z) \\ &\quad + 2|\log G(z)|^2 \operatorname{Re} \left\{ \overline{A(z)} B(z) \mathcal{L}[\log(\log G(z))] \right\} \\ &\quad + 2\operatorname{Re} \left\{ \overline{A(z) \log G(z)} \mathcal{L}[\log(f(z)h(\bar{z}))] \right\} + 2\operatorname{Re} \left\{ \overline{B(z)} C(f, h, G; z) \right\}, \end{aligned}$$

where

$$A(z) = \sum_{k=2}^p \lambda_k |z|^{2(k-2)} (k-1), \quad B(z) = \sum_{k=1}^p \lambda_k |z|^{2(k-1)}$$

and

$$C(f, h, G; z) = \frac{f'(z)}{f(z)} \frac{\overline{G_z(z)}}{\overline{G(z)}} - \frac{h'(\bar{z})}{h(\bar{z})} \frac{\overline{G_{\bar{z}}(z)}}{\overline{G(z)}}.$$

*Proof.* Taking the logarithm of

$$F(z) = f(z)h(\bar{z}) \prod_{k=1}^p G(z)^{\lambda_k |z|^{2(k-1)}},$$

and then differentiating both sides with respect to  $z$  and  $\bar{z}$ , respectively, we get

$$\begin{aligned} \frac{F_z(z)}{F(z)} &= \frac{f'(z)}{f(z)} + \bar{z}A(z) \log G(z) + B(z) \frac{G_z(z)}{G(z)}, \\ \frac{F_{\bar{z}}(z)}{F(z)} &= \frac{h'(\bar{z})}{h(\bar{z})} + zA(z) \log G(z) + B(z) \frac{G_{\bar{z}}(z)}{G(z)}. \end{aligned}$$

Hence we have

$$\begin{aligned} \left| \frac{F_z(z)}{F(z)} \right|^2 &= \left\{ \frac{f'(z)}{f(z)} + \bar{z}A(z) \log G(z) + B(z) \frac{G_z(z)}{G(z)} \right\} \cdot \left\{ \frac{\overline{f'(z)}}{\overline{f(z)}} + \overline{zA(z) \log G(z)} + \overline{B(z)} \frac{\overline{G_z(z)}}{\overline{G(z)}} \right\} \\ &= \left| \frac{f'(z)}{f(z)} \right|^2 + 2\operatorname{Re} \left\{ \frac{f'(z)}{f(z)} \left( \overline{zA(z) \log G(z)} + \overline{B(z)} \frac{\overline{G_z(z)}}{\overline{G(z)}} \right) \right\} \\ &\quad + 2\operatorname{Re} \left\{ \overline{zA(z)} B(z) \overline{\log G(z)} \frac{G_z(z)}{G(z)} \right\} + |z|^2 |A(z)|^2 |\log G(z)|^2 + |B(z)|^2 \left| \frac{G_z(z)}{G(z)} \right|^2 \end{aligned}$$

and

$$\begin{aligned} \left| \frac{F_{\bar{z}}(z)}{F(z)} \right|^2 &= \left| \frac{h'(\bar{z})}{h(\bar{z})} \right|^2 + 2\operatorname{Re} \left\{ \frac{h'(\bar{z})}{h(\bar{z})} \left( \overline{\bar{z}A(z) \log G(z)} + \overline{B(z)} \frac{\overline{G_{\bar{z}}(z)}}{\overline{G(z)}} \right) \right\} \\ &\quad + 2\operatorname{Re} \left\{ \overline{\bar{z}A(z)} B(z) \overline{\log G(z)} \frac{G_{\bar{z}}(z)}{G(z)} \right\} + |z|^2 |A(z)|^2 |\log G(z)|^2 + |B(z)|^2 \left| \frac{G_{\bar{z}}(z)}{G(z)} \right|^2. \end{aligned}$$

Then, it follows from  $J_{\log F}(z) = \frac{|F_z(z)|^2 - |F_{\bar{z}}(z)|^2}{|F(z)|^2}$  that

$$\begin{aligned}
& J_{\log F}(z) \\
&= \left| \frac{f'(z)}{f(z)} \right|^2 - \left| \frac{h'(\bar{z})}{h(\bar{z})} \right|^2 + |B(z)|^2 \left( \left| \frac{G_z(z)}{G(z)} \right|^2 - \left| \frac{G_{\bar{z}}(z)}{G(z)} \right|^2 \right) \\
&\quad + 2\operatorname{Re} \left\{ \overline{A(z)} B(z) \overline{\log G(z)} \left( z \frac{G_z(z)}{G(z)} - \bar{z} \frac{G_{\bar{z}}(z)}{G(z)} \right) \right\} \\
&\quad + 2\operatorname{Re} \left\{ \overline{A(z)} \left( \frac{zf'(z)}{f(z)} - \frac{\bar{z}h'(\bar{z})}{h(\bar{z})} \right) \overline{\log G(z)} \right\} + 2\operatorname{Re} \left\{ \overline{B(z)} C(f, h, G; z) \right\} \\
&= \left| \frac{f'(z)}{f(z)} \right|^2 - \left| \frac{h'(\bar{z})}{h(\bar{z})} \right|^2 + |B(z)|^2 \left( \left| \frac{G_z(z)}{G(z)} \right|^2 - \left| \frac{G_{\bar{z}}(z)}{G(z)} \right|^2 \right) \\
&\quad + 2|\log G(z)|^2 \operatorname{Re} \left\{ \overline{A(z)} B(z) \frac{zG_z(z) - \bar{z}G_{\bar{z}}(z)}{G(z) \log G(z)} \right\} \\
&\quad + 2\operatorname{Re} \left\{ \overline{A(z)} \left( \frac{zf'(z)}{f(z)} - \frac{\bar{z}h'(\bar{z})}{h(\bar{z})} \right) \overline{\log G(z)} \right\} + 2\operatorname{Re} \left\{ \overline{B(z)} C(f, h, G; z) \right\} \\
&= \left| \frac{f'(z)}{f(z)} \right|^2 - \left| \frac{h'(\bar{z})}{h(\bar{z})} \right|^2 + |B(z)|^2 J_{\log G}(z) + 2|\log G(z)|^2 \operatorname{Re} \left\{ \overline{A(z)} B(z) \mathcal{L}[\log(\log G(z))] \right\} \\
&\quad + 2\operatorname{Re} \left\{ \overline{A(z)} \log G(z) \mathcal{L}[\log(f(z)h(\bar{z}))] \right\} + 2\operatorname{Re} \left\{ \overline{B(z)} C(f, h, G; z) \right\}.
\end{aligned}$$

The proof is complete.  $\square$

**Corollary 2.** Suppose  $F \in \mathcal{L}_p\mathcal{H}(G)$  with  $F(z) = G(z)^{|z|^{2(p-1)}}$ , where  $p \geq 2$ . Then the Jacobian of  $\log F$ ,  $J_{\log F}$ , is given by

$$J_{\log F}(z) = |z|^{4(p-1)} J_{\log G}(z) + 2(p-1)|\log G(z)|^2 |z|^{2(2p-3)} \operatorname{Re} \{ \mathcal{L}[\log(\log G(z))] \}.$$

**Theorem 3.** Suppose that  $F \in \mathcal{L}_p\mathcal{H}(G)$  with

$$F(z) = f(z)h(\bar{z}) \prod_{k=1}^p G(z)^{\lambda_k |z|^{2(k-1)}},$$

the mapping  $\log G$  is starlike and orientation preserving,

$$\operatorname{Re} \left( \bar{z} \frac{f'(\bar{z})}{f(\bar{z})} \mathcal{L}[\log G(z)] \right) > 0,$$

$\frac{\bar{z}f'(\bar{z})}{f(\bar{z})} = \frac{zh'(z)}{h(z)}$ , and  $\sum_{k=1}^p \lambda_k \neq 0$  with  $\lambda_k \geq 0$  for all  $k \in \{1, \dots, p\}$ . Then  $\log F$  is orientation preserving and consequently locally univalent for  $z \in \mathbb{D} \setminus \{0\}$ .

*Proof.* It follows from

$$\frac{\bar{z}f'(\bar{z})}{f(\bar{z})} = \frac{zh'(z)}{h(z)}$$

that

$$\left| \frac{f'(z)}{f(z)} \right|^2 = \left| \frac{h'(\bar{z})}{h(\bar{z})} \right|^2 \quad \text{and} \quad \mathcal{L}[\log(f(z)h(\bar{z}))] = \frac{zf'(z)}{f(z)} - \frac{\bar{z}h'(\bar{z})}{h(\bar{z})} = 0.$$

Further, we get

$$\begin{aligned} 2\operatorname{Re} \left( \frac{f'(z)}{f(z)} \frac{\overline{G_z(z)}}{\overline{G(z)}} - \frac{h'(\bar{z})}{h(\bar{z})} \frac{\overline{G_{\bar{z}}(z)}}{\overline{G(z)}} \right) &= \frac{2}{|z|^2} \operatorname{Re} \left( \frac{zf'(z)}{f(z)} \frac{\bar{z}\overline{G_z(z)}}{\overline{G(z)}} - \frac{\bar{z}h'(\bar{z})}{h(\bar{z})} \frac{z\overline{G_{\bar{z}}(z)}}{\overline{G(z)}} \right) \\ &= \frac{2}{|z|^2} \operatorname{Re} \left( \frac{\bar{z}f'(\bar{z})}{f(\bar{z})} \mathcal{L}[\log G(z)] \right). \end{aligned}$$

Hence by Lemma 1, we see that

$$\begin{aligned} J_{\log F}(z) &= \left( \sum_{k=1}^p \lambda_k |z|^{2(k-1)} \right) \left\{ \left( \sum_{k=1}^p \lambda_k |z|^{2(k-1)} \right) J_{\log G}(z) + \frac{2}{|z|^2} \operatorname{Re} \left( \frac{\bar{z}f'(\bar{z})}{f(\bar{z})} \mathcal{L}[\log G] \right) \right. \\ &\quad \left. + 2 \left( \sum_{k=2}^p \lambda_k |z|^{2(k-2)} (k-1) \right) |\log G(z)|^2 \operatorname{Re}(\mathcal{L}[\log(\log G(z))]) \right\}. \end{aligned}$$

Since the assumptions imply that  $\operatorname{Re}(\mathcal{L}[\log(\log G)]) > 0$  for  $z \in \mathbb{D} \setminus \{0\}$ ,  $J_{\log G}(z) > 0$  and  $\sum_{k=1}^p \lambda_k |z|^{2(k-1)} > 0$  for  $z \in \mathbb{D} \setminus \{0\}$ , it follows that  $J_{\log F}(z) > 0$  for  $z \in \mathbb{D} \setminus \{0\}$ , that is,  $\log F$  is orientation preserving, and hence  $\log F$  is locally univalent for  $z \in \mathbb{D} \setminus \{0\}$ .  $\square$

**Corollary 3.** *Suppose that  $F \in \mathcal{L}_p \mathcal{H}(G)$  with  $F(z) = G(z)^{|z|^{2(p-1)}}$ , where  $p \geq 2$ , and that the mapping  $\log G$  is starlike and orientation preserving. Then  $\log F$  is orientation preserving and consequently locally univalent for  $z \in \mathbb{D} \setminus \{0\}$ .*

We define the linear operator  $\mathfrak{L}$  by

$$\mathfrak{L} = z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}}.$$

**Lemma 2.** *Suppose  $F \in \mathcal{L}_p \mathcal{H}$  with  $F(z) = \prod_{k=1}^p G_k(z)^{|z|^{2(k-1)}}$ . Then*

- (1)  $-i \frac{\partial \log F(re^{it})}{\partial t} = \sum_{k=1}^p |z|^{2(k-1)} L[\log G_k(z)],$
- (2)  $-\frac{\partial^2 \log F(re^{it})}{\partial^2 t} = \sum_{k=1}^p |z|^{2(k-1)} (\mathfrak{L}[\log G_k(z)] + z^2 (\log G_k(z))_{zz} + \bar{z}^2 (\log G_k(z))_{\bar{z}\bar{z}}).$

*Proof.* Since

$$\log F(z) = \sum_{k=1}^p |z|^{2(k-1)} \log G_k(z),$$

it follows that

$$\begin{aligned} (\log F(z))_z &= \sum_{k=2}^p (k-1) |z|^{2(k-2)} \bar{z} \log G_k(z) + \sum_{k=1}^p |z|^{2(k-1)} (\log G_k(z))_z, \\ (\log F(z))_{\bar{z}} &= \sum_{k=2}^p (k-1) |z|^{2(k-2)} z \log G_k(z) + \sum_{k=1}^p |z|^{2(k-1)} (\log G_k(z))_{\bar{z}} \end{aligned}$$

and

$$(\log F(z))_{z\bar{z}} = \sum_{k=2}^p (k-1)^2 |z|^{2(k-2)} \log G_k(z) + \sum_{k=2}^p (k-1) |z|^{2(k-2)} \mathfrak{L}[\log G_k(z)].$$

Therefore, by

$$(2.1) \quad \frac{\partial \log F(re^{it})}{\partial t} = iz(\log F(z))_z - i\bar{z}(\log F(z))_{\bar{z}},$$

we get

$$\begin{aligned} \frac{\partial \log F(re^{it})}{\partial t} &= iz \left( \sum_{k=2}^p (k-1) |z|^{2(k-2)} \bar{z} \log G_k(z) + \sum_{k=1}^p |z|^{2(k-1)} (\log G_k(z))_z \right) \\ &\quad - i\bar{z} \left( \sum_{k=2}^p (k-1) |z|^{2(k-2)} z \log G_k(z) + \sum_{k=1}^p |z|^{2(k-1)} (\log G_k(z))_{\bar{z}} \right) \\ &= i \sum_{k=1}^p |z|^{2(k-1)} L[\log G_k(z)], \end{aligned}$$

from which the proof of part (1) of the theorem follows.

Obviously, we know

$$\begin{aligned} &\mathfrak{L}[\log F(z)] - 2|z|^2 (\log F(z))_{z\bar{z}} \\ &= 2 \sum_{k=3}^p (3k - k^2 - 2) |z|^{2(k-1)} \log G_k(z) + \sum_{k=1}^p (3 - 2k) |z|^{2(k-1)} \mathfrak{L}[\log G_k(z)]. \end{aligned}$$

Upon differentiation we also have

$$\begin{aligned} (\log F(z))_{zz} &= \sum_{k=3}^p (k-1)(k-2) |z|^{2(k-3)} \bar{z}^2 \log G_k(z) + 2 \sum_{k=2}^p (k-1) |z|^{2(k-2)} \bar{z} (\log G_k(z))_z \\ &\quad + \sum_{k=1}^p |z|^{2(k-1)} (\log G_k(z))_{zz} \end{aligned}$$

and

$$\begin{aligned} (\log F(z))_{\bar{z}\bar{z}} &= \sum_{k=3}^p (k-1)(k-2) |z|^{2(k-3)} z^2 \log G_k(z) + 2 \sum_{k=2}^p (k-1) |z|^{2(k-2)} z (\log G_k(z))_{\bar{z}} \\ &\quad + \sum_{k=1}^p |z|^{2(k-1)} (\log G_k(z))_{\bar{z}\bar{z}}. \end{aligned}$$



Hence, we get

$$\begin{aligned}
& z^2(\log F(z))_{zz} + \bar{z}^2(\log F(z))_{\bar{z}\bar{z}} \\
&= 2 \sum_{k=3}^p (k-1)(k-2)|z|^{2(k-1)} \log G_k(z) + 2 \sum_{k=2}^p (k-1)|z|^{2(k-1)} \mathfrak{L}[\log G_k(z)] \\
&+ \sum_{k=1}^p |z|^{2(k-1)} (z^2(\log G_k(z))_{zz} + \bar{z}^2(\log G_k(z))_{\bar{z}\bar{z}}).
\end{aligned}$$

We infer that

$$\begin{aligned}
(2.2) \quad & -\frac{\partial^2 \log F(re^{it})}{\partial^2 t} \\
&= -\frac{\partial}{\partial t} [iz(\log F(z))_z - i\bar{z}(\log F(z))_{\bar{z}}] \\
&= \mathfrak{L}[\log F(z)] - 2|z|^2(\log F(z))_{z\bar{z}} + z^2(\log F(z))_{zz} + \bar{z}^2(\log F(z))_{\bar{z}\bar{z}} \\
&= \sum_{k=1}^p |z|^{2(k-1)} (\mathfrak{L}[\log G_k(z)] + z^2(\log G_k(z))_{zz} + \bar{z}^2(\log G_k(z))_{\bar{z}\bar{z}}),
\end{aligned}$$

from which the proof of part (2) follows.  $\square$

**Theorem 4.** Suppose that  $F \in \mathcal{L}_p\mathcal{H}(G)$  with

$$F(z) = f(z)h(z) \prod_{k=1}^p G(z)^{\lambda_k |z|^{2(k-1)}},$$

and suppose further that both  $f$  and  $h$  are constant functions,  $\log G$  and  $\log F$  are univalent,  $\mathcal{L}[\log G(z)] \neq 0$  and  $\sum_{k=1}^p \lambda_k |z|^{2(k-1)} \neq 0$  for  $z \in \mathbb{D} \setminus \{0\}$ . Then  $\log G$  is convex if and only if  $\log F$  is convex.

*Proof.* It follows from (2.1) and (2.2) in the proof of Lemma 2 that for  $z \in \mathbb{D} \setminus \{0\}$ ,

$$\begin{aligned}
& \frac{\partial}{\partial t} \left( \arg \frac{\partial \log F(re^{it})}{\partial t} \right) = \operatorname{Im} \left( \frac{\frac{\partial^2 \log F(re^{it})}{\partial t^2}}{\frac{\partial \log F(re^{it})}{\partial t}} \right) \\
&= \operatorname{Re} \left( \frac{\mathfrak{L}[\log F(z)] - 2|z|^2(\log F(z))_{z\bar{z}} + z^2(\log F(z))_{zz} + \bar{z}^2(\log F(z))_{\bar{z}\bar{z}}}{\mathcal{L}[\log F(z)]} \right) \\
&= \operatorname{Re} \left( \frac{\mathfrak{L}[\log G(z)] - 2|z|^2(\log G(z))_{z\bar{z}} + z^2(\log G(z))_{zz} + \bar{z}^2(\log G(z))_{\bar{z}\bar{z}}}{\mathcal{L}[\log G(z)]} \right) \\
&= \frac{\partial}{\partial t} \left( \arg \frac{\partial \log G(re^{it})}{\partial t} \right).
\end{aligned}$$

Then the assumptions imply that  $\log G$  is convex if and only if  $\log F$  is convex.  $\square$

### 3. GOODMAN AND SAFF'S CONJECTURE IN $\mathcal{L}_p\mathcal{H}(G)$

It is well-known that if an analytic function maps  $\mathbb{D}$  univalently onto a convex domain, then it also maps each concentric subdisk onto a convex domain (cf. [15]). Goodman and Saff ([17]) constructed an example of a function convex in the vertical direction whose restriction to the disk  $\mathbb{D}_r = \{z : |z| < r\}$  does not have that property for any radius  $r$  in the interval  $\sqrt{2} - 1 < r < 1$ . In the same paper, they conjectured that the radius  $\sqrt{2} - 1$  is best possible.

**Definition 1.** A domain  $D$  is *convex* in the direction  $e^{i\phi}$ , if for every fixed complex number  $z$ , the set  $D \cap \{z + te^{i\phi} : t \in \mathbb{R}\}$  is either connected or empty.

Let  $\mathcal{K}(\phi)$  (resp.  $\mathcal{K}_H(\phi)$ ) denote the class of all complex-valued analytic (resp. harmonic) univalent functions  $f$  in  $\mathbb{D}$  with  $f(\mathbb{D})$  convex in the direction  $e^{i\phi}$ . If  $f \in \mathcal{K}(\phi)$  (resp.  $\mathcal{K}_H(\phi)$ ) is such that  $f(\mathbb{D})$  is convex in every direction (i.e.  $f(\mathbb{D})$  is a convex domain), then in this case we say that  $f \in \mathcal{K}$  (resp.  $\mathcal{K}_H$ ). Ruscheweyh and Salinas [25, Theorem 1] ultimately succeeded in proving the Goodman-Saff conjecture by showing that if  $f \in \mathcal{K}_H(\phi)$  and  $r \in (0, \sqrt{2} - 1]$ , then one has  $f(rz) \in \mathcal{K}_H(\phi)$ . In particular, this gives the following.

**Theorem A.** ([25, Theorem 1]) *Let  $f \in \mathcal{K}_H$ . Then for any  $r \in (0, \sqrt{2} - 1]$ ,  $f(rz) \in \mathcal{K}_H$ .*

In view of the development of logharmonic mappings, it is interesting to ask whether the same conjecture holds for log-polyharmonic mappings. Our result is as follows.

**Theorem 5.** *Suppose that  $F \in \mathcal{L}_p\mathcal{H}(G)$  with*

$$F(z) = f(z)h(\bar{z}) \prod_{k=1}^p G(z)^{\lambda_k |z|^{2(k-1)}},$$

*where  $f$  and  $h$  are constant functions,  $\log F$  is univalent mapping,  $\log G$  is univalent and convex, and  $\mathcal{L}[\log G(z)] \neq 0$ ,  $\sum_{k=1}^p \lambda_k |z|^{2(k-1)} \neq 0$  for  $z \in \mathbb{D} \setminus \{0\}$ . Then  $\log F$  sends the subdisk  $\mathbb{D}_r$  onto a convex region for  $r \in (0, \sqrt{2} - 1]$ .*

*Proof.* Since  $\log G$  is harmonic and if we further assume that it is convex in  $0 < r \leq r_0 = \sqrt{2} - 1$ , then by Theorem 4 and Theorem A, we have that  $\log F$  is also convex in  $0 < r \leq r_0 = \sqrt{2} - 1$ . The proof is complete.  $\square$

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